Lecture 18

Physics 404

We considered the free electrons in a metal like Cu. Each Cu atom sits at a site in a lattice and gives up two electrons that are free to roam about the crystal. The electrons are spin-1/2 Fermions and have strongly overlapping wavefunctions. They must obey the Pauli exclusion principle, so that no two of them can occupy the same quantum state. At zero temperature they occupy all the single particle-in-a-box states starting from zero and going up to the highest occupied state, called the Fermi energy ($\mathrm{E_F}$).

If the electrons in a metal were classical particles in the dilute limit, we know from the last lecture that their total energy would be $U = \frac{3}{2}N\tau$ by equipartition of energy. Their heat capacity would be $C_V = \frac{\partial U}{\partial \tau}|_V = 3N/2$, which is substantial and temperature independent. However the electronic heat capacity of metals is observed to be much smaller than this and linear in temperature. The discrepancy is explained by the quantum statistical mechanics of the Fermi-Dirac distribution.

At zero temperature all of the single particle states are occupied with exactly one electron up to the most energetic state, beyond which all the states are empty. At finite temperature τ only those electrons within about τ of the chemical potential can be 'promoted' into empty states. All of the other electrons at lower energies cannot acquire enough energy from the reservoir to find an un-occupied state. Hence only a small fraction on the order of $\tau/\mathrm{E_F}$ of the electrons can actually absorb energy and find new states (For Cu $\mathrm{E_F} = 7$ eV and $\tau = 25$ meV at room temperature, so $\frac{\tau}{\mathrm{E_F}} = 0.004$). This greatly limits the heat capacity of the electron gas. As an estimate, the change in energy of the entire electron gas between $\tau = 0$ and temperature τ is roughly
$\Delta \mathrm{U} = (\#\ \text{of electrons promoted}) \times (\text{energy of each promoted electron}) \sim \left(\mathrm{N}\frac{\tau}{\mathrm{E_F}}\right) \times (\tau)$. The heat capacity is $C_V = \frac{\partial U}{\partial \tau}|_V = 2N\frac{\tau}{\mathrm{E_F}}$, which is much smaller than $3N/2$ and linear in temperature.

The Fermi energy can be calculated from the particle-in-a-box energy dependence on the triplet of integers (n_x, n_y, n_z), $\varepsilon_n = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_x^2 + n_x^2)$, where L is the side length of the box and the three integers go from 1 to infinity. The result is $\mathrm{E_F} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$, where $n = N/V$ is the concentration. Using similar arguments, one can calculate the total energy of the Fermi gas and find $\mathrm{U} = \frac{3}{5}\mathrm{NE_F}$. This is a substantial energy compared to the classical ideal gas ($U = \frac{3}{2}N\tau$), since $\mathrm{E_F} \gg \tau$.

We also discussed the Fermi gas properties of electrons in a white dwarf star. The situation for $\frac{\tau}{\mathrm{E_F}}$ is very similar to that of electrons in a metal, although the temperature and energy scales are much higher. The white dwarf resists gravitational collapse in part because of a quantum mechanical 'degeneracy pressure' arising from the Pauli exclusion principle and Fermi-Dirac statistics. The pressure can be found from $p = -\frac{\partial U}{\partial V}|_{\sigma,N} = \frac{2}{5}\mathrm{E_F}\frac{\mathrm{N}}{\mathrm{V}}$. This pressure scales as $\left(\frac{\mathrm{N}}{\mathrm{V}}\right)^{5/3}$, which means that the material will greatly increase its pressure as the concentration increases. This pressure, along with the thermal pressure of the gas at finite temperature, balance the gravitational force acting on the star.